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(23) 8.4 *Integrals of Rational Function (A)*

(1) A RETINAL FUNCTION is a quotient of two polynomials (1) الدالة النسبية هي خارج قسمة حدوديتين

For example $\frac{1}{x^2-4}$, $\frac{2x^2}{x(x-1)^2}$, $\frac{1}{x(x^2+x-1)}$ are retinal functions

But $\frac{1}{\sqrt{x}}$, $\frac{1}{e^x}$, $\frac{\ln x}{x}$, $\frac{|x-2|}{x^2}$, $\frac{\sin^2 x}{5x^2-7}$ are not retinal functions

(2) PARTIAL FRACTION

Any retinal function can be rewrite as sum of fractions with denominator of degree 1 or 2(hasn't real root) . These fraction is cold partial fraction

(2) الكسور الجزئية

إذا كان لدينا حدودية نسبية مقامها قابل للتحليل إلى عوامل من الدرجة الأولى أو الثانية فانه يمكن إعادة كتابتها على صورة مجموع عدة حدوديات نسبية مقاماتها من الدرجة الأولى أو الثانية (ليس لها جذور حقيقية)وتسمى هذه الحدوديات بالكسور الجزئية و تكون على الصور

$$\frac{A}{x+a} \; , \frac{B}{(x+a)^2} \; , \frac{Cx+D}{x^2+a^2}$$

To integrate a rational function it is usually necessary to using partial fraction

لإيجاد تكامل دالة نسبية يكون من الضروري وضعها على صورة مجموع من الكسور الجزئية

RATIONAL FUNCTION	PARTIAL FRACTION
$\frac{2x}{(x+2)(x-5)}$	$\frac{A}{x+2}+\frac{B}{x-5}$
$\frac{3x^2-1}{(x-7)(x+3)^2}$	$\frac{A}{x-7}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$
$\frac{x-1}{(x+4)(x^2+5)}$	$\frac{A}{x+4}+\frac{Bx+C}{x^2+5}$
$\frac{x+2}{(x-1)^2(x^2+7)}$	$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+7}$

Example 1 Evaluate the integral $\int \frac{2x^2-10x+15}{x^3-4x^2-5x}dx$ 42 December 2006

Solution

$x^3-4x^2-5x = x(x^2-4x-5) = x(x-5)(x+1)$

$\frac{2x^2-10x+15}{x^3-4x^2-5x} = \frac{2x^2-10x+15}{x(x-5)(x+1)} = \frac{A}{x}+\frac{B}{x-5}+\frac{C}{x+1}$

$A(x-5)(x+1)+Bx(x+1)+Cx(x-5) = 2x^2-10x+15$

at $x=0$ ⇨ $A(-5)(1)=15$ ⇨ $-5A=15$ ⇨ $\boxed{A=-3}$

at $x=5$ ⇨ $B(5)(6)=50-50+15$ ⇨ $30B=15$ ⇨ $\boxed{B=\frac{1}{2}}$

at $x=-1$ ⇨ $C(-1)(-6)=2-10(-1)+15$ $6C=27$ ⇨ $\boxed{C=\frac{9}{2}}$

$$I=\int \frac{2x^2-10x+15}{x^3-4x^2-5x}dx = \int\left(\frac{-3}{x}+\frac{\frac{1}{2}}{x-5}+\frac{\frac{9}{2}}{x+1}\right)dx = -3\ln|x|+\frac{1}{2}\ln|x-5|+\frac{9}{2}\ln|x+1|+c$$

Example 2 Evaluate the integral $\int \frac{x^2+2x+3}{x^4-1}dx$ 34 July 2004

Solution

$x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$

$\frac{x^2+2x+3}{x^4-1} = \frac{x^2+2x+3}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{Cx+D}{(x^2+1)}$

$A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)=x^2+2x+3$

at $x=-1$ ⇨ $B(-2)(2)=1-2+3$ ⇨ $-4B=2$ ⇨ $\boxed{B=\frac{-1}{2}}$

at $x=1$ ⇨ $A(2)(2)=1+2+3$ ⇨ $4A=6$ ⇨ $\boxed{A=\frac{3}{2}}$

t $x=0$ ⇨ $\left(\frac{3}{2}\right)(1)(1)+\left(-\frac{1}{2}\right)(-1)(1)+D(-1)(1)=3$

$\frac{3}{2}+\frac{1}{2}-D=3$ ⇨ $2-D=3$ ⇨ $\boxed{D=-1}$

at $x=2$ ⇨ $\left(\frac{3}{2}\right)(3)(5)+\left(\frac{-1}{2}\right)(1)(5)+(2c-1)(3)=4+4+3$

$\frac{45}{2}-\frac{5}{2}+6c-3=11$ ⇨ $20+6c-3=11$

$6c+17=11$ ⇨ $6c=-6$ ⇨ $\boxed{c=-1}$

$$\therefore I=\int \frac{x^2+2x+3}{x^4-1}dx = \int\left(\frac{\frac{3}{2}}{x-1}+\frac{-\frac{1}{2}}{x+1}+\frac{-x}{x^2+1}+\frac{-1}{x^2+1}\right)dx$$

$$I=\frac{3}{2}\ln|x-1|-\frac{1}{2}|x+1|-\frac{1}{2}\ln(x^2+1)-\tan^{-1}x+c$$

Example 3 Evaluate $\int \frac{x^2}{(x-1)(x^2+2x+1)}dx$

Solution

$(x-1)(x^2+2x+1) = (x-1)(x+1)(x+1) = (x-1)(x+1)^2$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$\therefore A(x+1)^2 + B(x-1)(x+1) + C(x-1) = x^2$

at $x = 1$ ⇨ $4A = 1$ ⇨ $\boxed{A = \frac{1}{4}}$

at $x = -1$ ⇨ $-2c = 1$ ⇨ $\boxed{C = \frac{-1}{2}}$

at $x = 0$ ⇨ $\left(\frac{1}{4}\right)(1) + B(-1)(1) + \left(\frac{-1}{2}\right)(-1) = 0$ ⇨ $\frac{3}{4} - B = 0$ ⇨ $\boxed{B = \frac{3}{4}}$

$$I = \int \frac{x^2}{(x-1)(x^2+2x+1)}dx = \int \left(\frac{\frac{1}{4}}{x-1} + \frac{\frac{3}{4}}{x+1} + \frac{-\frac{1}{2}}{(x+1)^2}\right)dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{\frac{1}{2}}{(x+1)} + c$$

Example 4 Evaluate $\int \frac{x^2-3x+2}{x^2-4x+3}dx$

Solution

$x^2 - 4x + 3 = (x-3)(x-1)$

$$x^2-4x+3 \overline{)\, x^2-3x+2} \quad (1)$$
$$\underline{-x^2+4x-3}$$
$$x-1$$

$$\frac{x^2-3x+2}{x^2-4x+3} = 1 + \frac{x-1}{x^2-4x+3} = 1 + \frac{x-1}{(x-3)(x-1)} = 1 + \frac{1}{(x-3)}$$

$$I = \int \frac{x^2-3x+2}{x^2-4x+3}dx = \int 1 + \frac{1}{(x-3)}\ dx = x + \ln|x-3| + c$$

تذكر

$$\int \frac{1}{x+a}dx = \ln|x+a| + c$$

$$\int \frac{1}{(x+a)^2}dx = \frac{-1}{x+a} + c$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Example 5 Evaluate $\int \frac{\cot x}{\sin x - 1} dx$ 4 December 1995

Solution

$$I = \int \frac{\cot x}{\sin x - 1} dx = \int \frac{\cos x}{\sin x\,(\sin x - 1)} dx$$

Let $u = \sin x \quad \Rightarrow \quad du = \cos x\ dx$

$$I = \int \frac{1}{u(u-1)} du$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{(u-1)}$$

$$A(u-1) + B\,u = 1$$

$at \quad x = 0 \quad \Rightarrow -A = 1 \quad \boxed{A = -1}$

$at \quad x = 1 \quad \Rightarrow \quad \boxed{B = 1}$

$$I = \int \left(-\frac{1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + c = -\ln|\sin x| + \ln|\sin x - 1| + c$$

Homework

1	Evaluate the integral $\int \frac{3x^2+1}{x^4-1}dx$	43 May 2007
2	Evaluate the integral $\int \frac{x^2-x+2}{x^3+2x^2+2x}dx$	41 July 2006
3	Evaluate the integral $\int \frac{x^3+8}{x^4+4x^2}dx$	37 June 2005
4	Evaluate the integral $\int \frac{5x^2+3}{x^4-1}dx$	40 May 2006
5	Evaluate the integral $\int \frac{5x^2+x-2}{(1+x^2)(3x-1)}dx$	
6	Evaluate the integral $\int \frac{x^2}{(x^2+1)^2}dx$	
7	Evaluate the integral $\int \frac{2x^2-3x+5}{x^4+5x^2+4}dx$	
8	Evaluate the following integral : ($3\frac{1}{2}$ points) $\int \frac{1}{(1+e^x)^2}dx$	50 Dec. 15, 2009
9	Evaluate the integral $\int \frac{x^3+x^2+x-1}{x^3-x^2+x-1}dx$	1 May 1994
10	Evaluate the integral $\int \frac{x+3x^3}{x^4-1}dx$	2 May 1995
11	Evaluate the integral $\int \frac{5x^2+x+6}{(x+1)(x^2+4)}dx$	3 August 1995

Homework

1	Evaluate the integral	$\int \frac{x^5+4x^3+11x}{(x^2+1)^2}dx$		4 December 1995
2	Evaluate the integral	$\int \frac{3x^2+11x+7}{(2x-1)(x^2+6x+10)}dx$		5 May 1996
3	Evaluate the integral	$\int \frac{x}{(x+1)(x^2+1)}dx$		6 July 1996
4	Evaluate the integral	$\int \frac{x^3+1}{x^3-1}dx$		7 November 1996
5	Evaluate the integral	$\int \frac{2+3x}{x^4+x^2}dx$		8 May 1997
6	Evaluate the integral	$\int \frac{3x+5}{x^3-4x^2+x-4}dx$		10 August 1997
7	Evaluate the integral	$\int \frac{4}{1+x-x^2-x^3}dx$		11 December 1997
8	Evaluate the integral	$\int \frac{2}{1+x+x^2+x^3}dx$		12 December 1997
9	Evaluate the integral	$\int \frac{6x^2-3x+1}{(4x+1)(x^2+1)}dx$		13 May 1998
10	Evaluate the integral	$\int \frac{x^2-2x+1}{x^4+2x^2+1}dx$		15 December 1998
11	Evaluate the integral	$\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}dx$		18 December 1999
12	Evaluate the integral	$\int \frac{2x^2+3}{x^3+x^2+4x+4}dx$		19 May 2000
13	Evaluate the integral	$\int \frac{x^3-1}{x^2+x-6}dx$		20 April 2000
14	Evaluate the integral	$\int \frac{2x^2+3}{x^3+x^2+4x+4}dx$		21 May 2000
15	Evaluate	$\int \frac{dx}{1+\sin x-\cos x}$	(4 pts)	41 14 January 2012